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EXOTIC FOLIATIONS ON SURFACES BUNDLES OVER THE CIRCLE

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Abstract

We construct an exotic foliation on hyperbolic bundles of genus 1, whose fiber is a punctured torus. We will use specially the method used in [4].

1. Introduction

In [3], Ghys and Sergiescu proved an important result of classification for nonsingular foliations without compact leaf on hyperbolic bundles of genus 1. They proved that any class $C^r (r \geq 2)$ transversely oriented foliation without compact leaf is conjugated to a model foliation built in [3]. Dathe and Tarquini [2] proved the optimality of this result by building exotic foliations in class C^0 and C^1. All these results lead with the compact case meaning on compact bundles. This motivates us to study

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foliations on open hyperbolic bundles. Hatcher [4] built a foliation without compact leaf on hyperbolic compact bundle with boundary. Taking inspiration with Hatcher work, we build smooth exotic foliations on any genus 1 open hyperbolic bundle proving that Ghys and Sergiescu result cannot be extended in the non-compact case. So, the compacity is essential in this classification. In the first part of this paper, we will recall some essential tools for the construction. The second part leads with how homotopies between foliations on $T \times [0, 1]$, where T is the punctured torus are built, which is the first step to the construction of exotic foliation, which is the last part of this work.

2. Preliminaries

2.1. Integrable homotopies

Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two codimension q foliations on a differentiable manifold M. A class C^r integrable homotopy between these foliations is a class C^r codimension q foliation $\mathcal{H}$ on $M \times [0, 1]$ transverse to $M \times \{t\}$ for $0 \leq t \leq 1$, such that $\mathcal{H}|M \times \{0\} = \mathcal{F}_0$ and $\mathcal{H}|M \times \{1\} = \mathcal{F}_1$.

Here homotopy can be seen as a deformation of $\mathcal{F}_0 = \mathcal{H}_0$ and $\mathcal{F}_1 = \mathcal{H}_1$ throughout intermediate foliations $\mathcal{H}_t$ induced by $\mathcal{H}$ on $M \times \{t\}$.

Example 1. Let ω_0 and ω_1 be two nonsingular closed 1-forms on a differentiable manifold M transverse to a dimension one foliation $\mathcal{L}$. Here transversality means that, if X is a nonsingular vector field tangent to $\mathcal{L}$, then $\omega_0(X) > 0$ and $\omega_1(X) > 0$ anywhere. According to Frobenius theorem, these two forms, respectively, define two codimension 1 transversely oriented foliations $\mathcal{F}_0$ and $\mathcal{F}_1$, whose transverse orientation is given by the orientation of $\mathcal{L}$. Suppose that the two forms are cohomological, meaning that $\omega_1 - \omega_0 = df$ for some class C^∞ function f on M. Define the one form Ω on $M \times [0, 1]$ as

$$\Omega_{(x,t)} = t\omega_{1x} + (1-t)\omega_{0x} + f(x,t)dt(x,t), \text{ for } (x,t) \in M \times [0,1].$$

Thus,

$$d\Omega = dt \wedge (\omega_1 - \omega_0) + df \wedge dt = dt \wedge df - dt \wedge df = 0.$$

The vector field X defined above can be viewed as a vector field on $M \times [0, 1]$ independent from t. So we have $\Omega(X) > 0$ everywhere. Ω is a closed nonsingular one form on $M \times [0, 1]$. It defines a codimension 1 foliation $\mathcal{H}$ transverse to $M \times [0, 1]$ for $0 \leq t \leq 1$, and $\Omega | M \times \{0\} = \omega_0$ and $\Omega | M \times \{1\} = \omega_1$. The foliation $\mathcal{H}$ is an integrable homotopy between $\mathcal{F}_0$ and $\mathcal{F}_1$.

2.2. Train tracks

Definition 1. A train track τ on a surface M is a compact subspace of M homeomorphic to a graph without boundary, whose edges are differentiable segments of M, tangent to each other at vertices. The vertices are called *switches*.

We said that a train track is oriented, if the tangent space at any point is oriented.

We suppose that the connected components of the complement of τ are homeomorphic to discs, whose boundary is not smooth.

A suitable neighborhood $V(\tau)$ of a train track τ is a thickening, whose boundary has singularities of punch type; this thickening has the same number of singularities at the boundary as the train track has vertices. Such a neighborhood is obtained, for example, by gluing through the sides a rectangle for each edge of the train track. A foliation by ties of $V(\tau)$ is a foliation of $V(\tau)$, transverse to the boundary, and whose leaves go from boundary to boundary. Collapsing the leaves of the foliation by ties, we obtained a train track.

When we have a suitable neighborhood V of a train track τ endowed with a system of ties, a longitudinal foliation is a foliation of V, whose leaves are including the boundary and without singularities inside.

If the connected components of the complement of τ are homeomorphic to discs, whose boundary is not smooth, so for $M \setminus V$. If we have a longitudinal foliation of V, we can collapse each connected component of the complement of V to get at most one saddle singularity. We obtain a foliation of the manifold M with a singularity in the center of any component of the complement of V except for bigoni, which collapse without showing any saddle singularity. The number of separators of the saddle point is equal to the number of punches on the boundary of the connected component of the complement of V.

Definition 2. A foliation $\mathcal{F}$ is carried by a train track τ, if we have

(1) a suitable neighborhood V;

(2) a foliation by ties $\mathcal{T}$ of V;

(3) a longitudinal foliation $\mathcal{F}_1$ of V;

satisfying

(a) the train track obtained by collapsing the ties is τ;

(b) $\mathcal{F}$ is obtained from $\mathcal{F}_1$ by collapsing the connected components of the complement of V;

(c) $\mathcal{T}$ and $\mathcal{F}_1$ are transverse.

If τ carries $\mathcal{F}$, each leaf of $\mathcal{F}$ is obtained from a leaf of $\mathcal{F}_1$.

A system of weight μ for τ is an assignment of a number $\mu(a)$ in $\mathbb{R}_+^*$ to each edge a of τ; these weights satisfy the compatibility condition at the switches: The sum of the outgoing weights at the switch is equal to the sum of ingoing weights. We say that a train track is recurrent, if it has a system of weights. If we have a train track endowed with a system of weights μ, then we can replace each edge by a foliated rectangle of larger $\mu(a)$. According to the compatibility, the rectangles are glued to give a measured foliation carried by τ.

2.2.1. Transversely affine laminations

Let M be a manifold and $\widetilde{M}$ be its universal covering. A transversely affine structure for a foliation $\mathcal{F}$ of M (with singularity) attributes to each path in M transverse to $\mathcal{F}$, an affine structure invariant by the homotopy of paths transverse to the leaves.

Let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal covering $\widetilde{M}$ of M. A transversely affine structure of $\mathcal{F}$ is lifted to a transversely affine structure for $\widetilde{\mathcal{F}}$. As $\widetilde{M}$ is simply connected, the affine transverse structure on $\widetilde{\mathcal{F}}$ determines a unique transversely euclidian structure up to multiplication by a scalar number obtained by choosing a measure for the transverse segment and which can extend to the others transverse segments. Any deck transformation of $\widetilde{M}$ sends a transversely euclidian structure of $\widetilde{\mathcal{F}}$ on a scalar multiple transversely euclidian structure of the above one.

Conversely, a class of projection of transversely euclidian structures of $\widetilde{\mathcal{F}}$ for which the deck transformations act as multiplications by scalars determine a transversely affine structure of $\mathcal{F}$. We can then give the next definition of transversely affine structure for a lamination.

Definition 3. Let L be a lamination of M and $\widetilde{L}$ its lift to $\widetilde{M}$. We say that L is a transversely affine lamination, if $\widetilde{L}$ is a transversely measured lamination such that deck transformations act as multiplications by scalar.

3. Homotopy Between Foliations on $T \times [0, 1]$

To built foliations without compact leaf on $T \times I$, we will consider a train track (see Figure 4 in [4]) on the universal covering $\widetilde{T}$ of the torus T private to an open disc, which is $\mathbb{R}^2$ private of a neighborhood of $\mathbb{Z}^2$ and assign weights to the train track such that right translations by one unit multiply the weight by 2 while up translations multiply by a parameter $\mu > 0$. These weights define a measured lamination $\widetilde{L}(\mu, t)$ on $\widetilde{T}$ invariant by deck transformations with the measure multiply by a scalar hence lifts a transversely affine lamination $L(\mu, t)$ on T. Collapsing the supplementary of the lamination $L(\mu, t)$, one can obtains a transversely foliation $\mathcal{F}(\mu, t)$ on T transverse to ∂T with a unique saddle singularity. We can consider the supplementary of the lamination as a quadrilateral and collapsing it, we get a saddle singular point. Two particular cases are the case $t = 0$ and the case where t and μ verify the condition $2 + (1 - \mu)t = \mu t$.

If $t = 0$, the lamination $L(\mu, 0)$ contained a unique closed leaf, which is a circle projection of a line of slope 0. The other leaves form a band of parallel half-lines starting at ∂T and spiralling into the circle leaf from below. If we choose $(\mu, t) = (\mu_1, t_1)$ satisfying $2 + (1 - \mu)t = \mu t$, then the lamination $\widetilde{L}(\mu_1, t_1)$ has a single closed leaf, which is a circle of slope 1 and all the other leaves spiralling in to this circle from below.

Note that if we fix μ_1 and let t increase, we are transferring sheets of leaves of thickness t from one pair of opposite sides of the complementary quadrilateral of $L(\mu, t)$ to the other pair of opposite sides, as indicated in [4]. So, we can construct a measured lamination in $\widetilde{T} \times [0, 1]$, which meets $T \times \{0\}$ in $L(\mu_1, 0)$ and $T \times \{1\}$ in $L(\mu_1, t_1)$ with a layer of saddles achieving the surgery of $L(\mu_1, 0)$ to $L(\mu_1, t_1)$. So, we obtain foliation on $T \times [0, 1]$, which we denote by $\mathcal{F}_{t_1}\{(1, 0), (1, 1)\}$, the vectors $(1, 0)$ and $(1, 1)$ corresponding to the limit cycles of the boundary foliations $\mathcal{F}(\mu_1, 0)$ in $T \times \{0\}$ and $\mathcal{F}(\mu_1, t_1)$ in $T \times \{1\}$, oriented by the direction of spiralling. This foliation meets each slice $T \times s$ in foliation transverse to $\partial T \times \{s\}$ with a single saddle singularity. So, this foliation is a homotopy between foliations $\mathcal{F}(\mu_1, 0)$ in $T \times \{0\}$ and $\mathcal{F}(\mu_1, t_1)$ in $T \times \{1\}$.

Example 2. We can construct examples of these foliations by considering suitable linear map in $SL(2, \mathbb{Z})$ to T. We obtain foliations $\mathcal{F}_{t_1}\{(p, q), (r, s)\}$ in $T \times [0, 1]$ meeting $T \times \{0\}$ and $T \times \{1\}$ in foliations (each with a single saddle singularity), which spiral in from the right to oriented limit cycles with slopes q / p and s / r provided $ps - qr = 1$.

4. Exotic Foliations on Hyperbolic Bundles of Genus 1

4.1. Punctured torus bundles over the circle

Set $T = T^2 \setminus \{0\}$ as the punctured torus and $\widetilde{T}$ as the plan $\mathbb{R}^2$ private with $\mathbb{Z}^2$. Consider $A \in SL(2, \mathbb{Z})$ be a hyperbolic automorphism of $\widetilde{T}$, it induces a diffeomorphism on T denoted ϕ. Consider in T, the foliations by parallel lines to the eigendirections of A. These foliations have dense leaves. We define the punctured torus bundle

$$M_\phi = T \times [0, 1] / (x, 0) \sim (\phi(x), 1).$$

The product foliation on $T \times [0, 1]$ descends to the quotient on M_ϕ to a foliation, we call model foliation. We obtain two model foliations on M_ϕ. A foliation on M_ϕ is called *exotic*, if it is not conjugated to any model foliation on M_ϕ.

We will prove the following theorem:

Theorem 1. *There is on any punctured torus bundle an exotic class $C^r (\geq 2)$ foliation without compact leaf.*

4.2. Proof

Step 1. Construction

Let $T \hookrightarrow M_\phi \to S^1$ be the bundle, whose fiber T is the torus private with an open disc and monodromy $\phi \in SL(2, \mathbb{Z})$ with $\operatorname{tr}\phi > 2$.

We consider for ϕ, the sequence $(p_i, q_i) \in \mathbb{Z}^2$ with $i = 0, \ldots, n$; $p_i q_{i+1} - q_i p_{i+1} = 1$ and $\phi(p_0, q_0) = (p_n, q_n)$ corresponding to the edge paths in the diagram of $SL(2, \mathbb{Z})$. For each i, we obtain the foliation $\mathcal{F}_{t_1}\{(p_i, q_i), (p_{i+1}, q_{i+1})\}$ in $T \times [0, 1]$, which is an homotopoy between foliation on the boundary foliations as above. $\mathcal{F}_{t_1}\{(p_i, q_i), (p_{i+1}, q_{i+1})\}$ and $\mathcal{F}_{t_1}\{(p_{i+1}, q_{i+1}), (p_{i+2}, q_{i+2})\}$ are gluing together because they have the same limit cycle for the foliation in the boundary component to be glued. The relation $\phi(p_0, q_0) = (p_n, q_n)$ allows us to glue the last components using ϕ. We require the gluing to preserve the transverse measure at a fixed lift of ∂T to $\widetilde{T}$. So, we obtain a transversely affine foliation on M_ϕ, which meets the boundary in a foliation with a transverse measure. We denote $\mathcal{F}$ be the resulting foliation on M_ϕ, which has no compact leaf (see [4]).

Step 2. Desingularization

Remark 1. Let T be the torus private with an open disc. The Euler number of T is $\xi(T) = -1$. According to Hopf theorem, if $\mathcal{F}$ is a singular foliation on T with a single saddle singularity s inside T, it has four separators. Indeed, $\xi(T) = \text{ind}(s) = 1 - \frac{4}{2} = -1$. Also, if the singularity s is at the boundary and the foliation transverse to ∂T, then s has three separators. Indeed, $\xi(T) = \frac{1-3}{2} = -1$.

The foliation $\mathcal{F}$ is a singular foliation with a singularity homeomorphic to a circle. As the single singularity in a fiber is a saddle singularity with an even number of separators, we can do the desingularization of the foliation $\mathcal{F}$ to get a smooth one in M_ϕ. Let C be the singular circle of the foliation $\mathcal{F}$. Consider V be a closed tubular neighborhood of C in M_ϕ such that $\mathcal{F}|\partial V$ is a union of product foliation of smooth foliation of torus, whose leaves are isotopic to $\partial V \cap (T \times \{t\}) / (x, o) \sim (\phi(x), 1) (t \in [0, 1])$.

By attaching copies of the product foliation $\{D^2 \times \{x\};\ x \in S^1\}$ of $D^2 \times S^1$ to $\mathcal{F}|(M_\phi - \operatorname{int} V)$ along the leaves of $\partial D^2 \times S^1$ and ∂V, we obtain a C^∞ foliation denoted also $\mathcal{F}$. Removing the boundary of T and as the foliation on the fibers is transverse to the boundary, one can obtains a foliation $\mathcal{G}$ on the once-punctured torus bundle without compact leaf.

Step 3

Lemma 1. *The foliation $\mathcal{G}$ is exotic.*

Proof. To prove that $\mathcal{G}$ is an exotic foliation, we will consider how it intersects the fibers of M_ϕ. The trace of the foliation $\mathcal{G}$ varies from fiber to fiber. Also, it is not a product foliation unlike the model foliations. The trace of the foliation $\mathcal{G}$ on the boundary of $T \times [0, 1]$ contains a compact leaf and the other leaves spiralling in to this compact leaf. This is not true in the case of model foliations, whose trace in the fiber is regular minimal foliations. So, the traces of these foliations on the fiber are not conjugated. Hence, the foliation $\mathcal{G}$ is exotic. □

Remark 2. As the foliation on T is transverse to the boundary, one separator of the singularity is transverse to the boundary. We can do a Whithead operation by deleting this separator, hence we carry the singularity on the boundary. Removing the boundary, the singularity is deleted. So, we obtain a nonsingular foliation on T.

Example 3. Let $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$, i.e., $\phi(x, y) = (2x + y, x + y)$.

One obtains the sequence $(p_i, q_i) \in \mathbb{Z}^2$ such that $(p_0, q_0) = (1, 0)$ and $(p_7, q_7) = (2, 1) = \phi(1, 0)$ verifies $p_i q_{i+1} - q_i p_{i+1} = 1$. We have the following sequence from the diagram of $SL(2, \mathbb{Z})$:

$$(1, 0);\ (-2, 1);\ (1, -1);\ (-1, 2);\ (0, -1);\ (1, 2);\ (-1, -1);\ (2, 1).$$

Considering the foliations $\mathcal{F}_{t_i}\{(p_i, q_i), (p_{i+1}, q_{i+1})\}$ in $T \times I$, one can obtains an exotic foliation on M_ϕ.

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